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1987 J. Phys. A: Math. Gen. 20 5707

(http://iopscience.iop.org/0305-4470/20/16/043)

J. Phys. A: Math Gen. **20** (1987) 5707-5717. Printed in the UK

Operator content of the *ADE* lattice models

Vincent Pasquier
Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 16 February 1987

Abstract. We compute the scaling dimensions of order parameters in the *ADE* lattice models. Some connection is made between the lattice algebra and the operator algebra in conformal invariant theories.

1. Introduction

In a recent work (Pasquier 1987a) the author defined lattice statistical models which led him to conjecture a classification for unitary conformal theories with a central charge smaller than or equal to one (Friedan *et al* 1984). In these models it was not difficult to guess the expression of order parameters generalising results previously obtained by Andrews *et al* (1984) and Huse (1984) in the restricted solid-on-solid (RSOS) model case. It remained to compute their critical exponents which is the aim of this paper. To obtain them, we re-express the two-point correlation function of magnetic operators in terms of correlation functions which can be evaluated in the continuum limit. This method generalises to these models the one used by den Nijs (1983) to compute the magnetic exponents of the Potts model.

In § 1 we describe the models and introduce the basic techniques required to study them. In § 2 we compute the critical exponents and identify the order parameters. In § 3 we make a connection between the operator algebra of lattice models and the operator algebra in conformal theories. This enables us to conjecture about the lattice models.

1.1. The models

The models are characterised by a square lattice on a cylinder and a Coxeter diagram (figure 1). The square lattice rotated by 45° has N rows and M columns (figure 2). The rows are numbered from 0 to N. Similarly, the columns are numbered from 0 to $M-1$, the column numbered M being identical with that numbered 0. Each point of the Coxeter diagram has a height marked on it. Each site of the lattice is assigned an arbitrary height with the restriction that two heights on neighbouring sites are also heights on neighbouring points of the Coxeter diagram. Each assignment of heights induces a decomposition of the lattice into two interpenetrating (even and odd) sublattices such that all heights of a given sublattice have the same parity.

In a transfer matrix approach we define the Hilbert space on which operators act: consider a column of the lattice as in figure 3 and label the sites by i (i will sometimes be called the level). A base for states will be a set of heights $|\sigma\rangle = |\dots\sigma_{-K}\sigma_{-K+1}\dots\sigma_N\dots\rangle$ such that the two successive heights are constrained to be

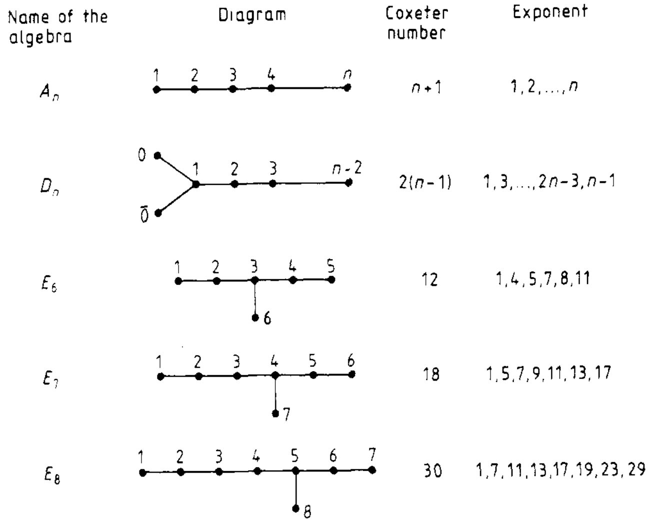

Figure 1. The Coxeter diagram.

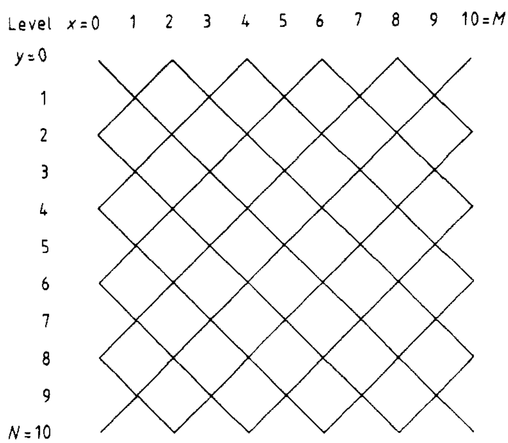

Figure 2. The square lattice.

linked on the Coxeter diagram (figure 1). It can also be viewed as an infinitely long path on the diagram (figure 1). Far from the origin, paths are assumed to obey certain periodicity conditions characterising the vacuum.

1.2. Operator algebra

We define $\mathscr{D}_{KN}$ as the algebra of matrices which act on heights σ_1, $K \leqslant \sigma_1 \leqslant N$ and let σ_K, σ_N remain unchanged:

$$d|\ldots\sigma_K\sigma_{K+1}\ldots\sigma_N\ldots\rangle = \sum_{\sigma'_{K+1}\ldots\sigma'_{N-1}} d_{\sigma_K\sigma_{K+1}\ldots\sigma_N,\sigma_K\sigma'_{K+1}\ldots\sigma'_{N-1}\sigma_N}|\ldots\sigma_K\sigma'_{K+1}\ldots\sigma_N\ldots\rangle. \tag{1}$$

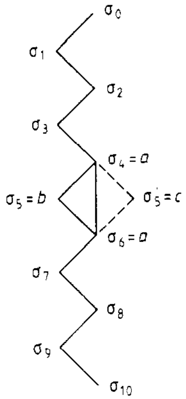

Figure 3. A column of the lattice $\tilde{e}_5$ is represented at level 5. It induces a bar between σ_4 and σ_6.

A base of $\mathscr{D}_{KN}$ (as was first discussed by Ocneanu (1986)) can be built as follows. Consider two paths α, β of length n going from identical heights at level K and ending at identical heights at level N. Define an operator $f_{\alpha\beta}$ in $\mathscr{D}_{KN}$:

$$\begin{aligned} f_{\alpha\beta}|\beta\rangle &= |\alpha\rangle \\ f_{\alpha\beta}|\gamma\rangle &= 0 \qquad \text{if} \qquad \gamma \neq \beta. \end{aligned} \tag{2}$$

Clearly the set $\{f_{\alpha\beta}\}$ indexed by paths starting and ending at the same height makes a base of $\mathscr{D}_{KN}$.

The total operator algebra is the reunion of all $\mathscr{D}_{KN}$

$$\mathscr{D} = \overline{U\mathscr{D}_{KN}}. \tag{3}$$

It can be convenient to define $\mathscr{D}_{KN}$ as the set of matrices that commute with $\mathscr{D}_{-\infty K}$ and $\mathscr{D}_{N+\infty}$

$$\mathscr{D}_{KN} = \mathscr{D}'_{-\infty K} \cap \mathscr{D}'_{N+\infty} \tag{4}$$

where $\mathscr{D}'$ means the commutant of $\mathscr{D}$. $\mathscr{D}$ naturally splits into two components:

$$\mathscr{D} = \mathscr{D}^+ \oplus \mathscr{D}^- \tag{5}$$

where $\mathscr{D}^+$ acts only on paths such that $\sigma_i - i$ is even and $\mathscr{D}^-$ on paths such that $\sigma_i - i$ is odd.

1.3. Trace

In an earlier paper (Pasquier 1987a) we defined the trace of a projector as the dimension of its image space divided by the dimension of the total Hilbert space in the infinite lattice limit. The result can be described as follows (this trace was also considered by Ocneanu (1986)).

To the diagram of figure 1 attach its incidence matrix C acting on vector components labelled by heights on the diagram v_j

$$(C \quad v)_i = \sum_{j \sim i} v_j \tag{6}$$

where $\Sigma_{j\sim i}$ means that the summation is on all points j neighbouring i on the diagram. Let S_i be the vector with the largest eigenvalue β of C; all components of S_i are positive by the Perron–Frobenius theorem. Normalise S_i such that

$$\sum_i S_i^2 = 1 \qquad \text{if} \qquad P_{\sigma_K \ldots \sigma_N}$$

is the projector of $\mathscr{D}_{KN}$ defined by

$$P_{\sigma_K \ldots \sigma_N} | \ldots \sigma_K \sigma_{K+1} \ldots \sigma_N \ldots \rangle = | \ldots \sigma_K \sigma_{K+1} \ldots \sigma_N \ldots \rangle = 0 \qquad \text{on any other path.} \tag{7}$$

Then

$$\operatorname{Tr} P_{\sigma_K \ldots \sigma_N} = \beta^{-(N-K)} S_K S_N. \tag{8}$$

Since

$$P_{\sigma_K \ldots \sigma_N} = \sum_{\sigma_{N+1} \sim \sigma_N} P_{\sigma_K \ldots \sigma_N \sigma_{N+1}}$$

both projectors should have the same trace which results from

$$\sum_{\sigma_{N+1} \sim \sigma_N} S_{\sigma_{N+1}} = \beta S_{\sigma_N}$$

$\operatorname{Tr} \mathbb{1} = 1$ due to the normalisation of S_i.

The physical interpretation of this trace is clarified by noting that $\operatorname{Tr} P_{\sigma_K}$ is the expectation value $\langle P_{\sigma_K} \rangle$ computed by Andrews *et al* (1984) at the critical point in the RSOS models (the diagram in figure 1 is then that of A_{r-1} where r is their parameter).

1.4. *Temperley–Lieb–Jones* (TLJ) *algebra*

Consider two algebras $\mathscr{A} \subseteq \mathscr{B}$. There is a projector $E_{\mathscr{A}}$ from $\mathscr{B}$ to $\mathscr{A}$ defined by the property (Jones 1983)

$$\operatorname{Tr} E(x) y = \operatorname{Tr}(xy) \qquad \forall x \in \mathscr{B},\ \forall y \in \mathscr{A}. \tag{9}$$

E is the orthogonal and has the property

$$E(axb) = aE(x)b \qquad \text{for} \qquad a, b \in \mathscr{A}. \tag{10}$$

Applying this definition to $\mathscr{A} = \mathscr{D}_{K,N}$, $\mathscr{B} = \mathscr{D}_{K,N+1}$ it is possible to build a projector e_{n+1} of $\mathscr{D}_{K,N+2} \cap \mathscr{D}'_{K,N}$ such that for $x \in \mathscr{D}_{K,N+1} \cap \mathscr{D}'_{K,N}$ one has

$$e_{n+1} x e_{n+1} = E(x) e_{n+1} \tag{11}$$

and whose expression is given by

$$e_{n+1} | \ldots \sigma_n \sigma_{n+1} \sigma_{n+2} \ldots \rangle = \delta(\sigma, \sigma_{n+2}) \beta^{-1} \sum_{\sigma'_{n+1} \sim \sigma_n} \frac{(S_{\sigma'_{n+1}} S_{\sigma_{n+1}})^{1/2}}{S_{\sigma_n}} | \ldots \sigma_n \sigma'_{n+1} \sigma_n \ldots \rangle. \tag{12}$$

The set of matrices so built obey the Temperley–Lieb (1971) relations together with the Jones trace condition:

$$\begin{aligned} & e_n^2 = e_n \\ & e_n e_{n\pm 1} e_n = \beta^{-2} e_n \\ & e_n e_m = e_m e_n \qquad \text{for } |n-m| \geqslant 2 \\ & \operatorname{Tr} e_{n_1} e_{n_2} \ldots e_{n_k} = \beta^{-2k} \qquad \text{for } n_1 > n_2 > \ldots > n_k. \end{aligned} \tag{13}$$

We shall call it the TLJ algebra. This representation was independently obtained by Ocneanu (1986) in a different context and observed to occur in the RSOS models by Akutsu *et al* (1986).

1.5. Partition function

Using the TLJ algebra, one can define a partition function

$$Z = \mathrm{Tr}(VW)^m \tag{14}$$

with

$$\begin{aligned} V &= \ldots(1+x_1e_{2n+1})(1+x_1e_{2n+3})\ldots \\ W &= \ldots(x_2+e_{2n})(x_2+e_{2n+2})\ldots. \end{aligned} \tag{15}$$

This depends on two parameters x_1, x_2. The model is self-dual on the line $x_1x_2 = 1$ (Baxter 1982). In what follows, we shall restrict ourselves to this line and take $x_1 = \beta$, $x_2 = \beta^{-1}$.

2. Critical exponents

To compute the critical exponents of magnetic operators, we shall re-express the correlation function of these operators computed on the self-dual line in terms of the correlation functions which can be computed in the continuum limit. For this, we introduce the following models called BC-SOS models (van Beijeren 1977). Heights take integer values. Each site of the lattice is assigned an arbitrary height with the restriction that their values on neighbouring sites differ by ± 1. The equation analogous to (12) is

$$e_{n+1}|\ldots\sigma_n\sigma_{n+1}\sigma_{n+2}\ldots\rangle = \delta(\sigma_n,\sigma_{n+2})\beta^{-1}\sum_{\sigma'_{n+1}=\sigma_n\pm 1} Z^{(\sigma_{n+1}+\sigma'_{n+1}-2\sigma_n)/2}|\ldots\sigma_n\sigma'_{n+1}\sigma_{n+2}\ldots\rangle. \tag{16}$$

Z is a complex number such that

$$Z+Z^{-1} = \beta. \tag{17}$$

Later, the value of Z will be adjusted so that β is equal to the value appearing in (12). The equation analogous to (7) is

$$\mathrm{Tr}\, P_{\sigma_K\ldots\sigma_N} = \beta^{-(N-K)}Z^{(\sigma_K+\sigma_N)}. \tag{18}$$

Given a Coxeter diagram (figure 1), let us define the following operators in $\mathscr{D}_{NN}$:

$$\varphi_m(N) = \sum_a V_m^a \frac{P_{\sigma_N=a}}{S^a} \tag{19}$$

where V_m^a is an eigenvector with eigenvalue $\lambda_m = 2\cos(m\pi/h)$ of the incidence matrix. In the BC-SOS model, the corresponding expression for V_m^a is either Z^{ma} or Z^{-ma} so that φ_m becomes

$$\varphi_m^1 = \sum_{a\in\mathbb{Z}} Z^{(m-1)a}P_{\sigma_N=a}$$

or (20)

$$\varphi_m^2 = \sum_{a\in\mathbb{Z}} Z^{(-m-1)a}P_{\sigma_N=a}.$$

The correlation function of several φ_m is defined to be

$$\langle\Phi_m(n_1N_1)\varphi_m(n_2N_2)\ldots\rangle = Z^{-1}\,\mathrm{Tr}\{\underbrace{UVU\ldots}_{n_1\text{ terms}}\varphi_m(N_1)\underbrace{UV\ldots}_{n_2-n_1\text{ terms}}\varphi_m(N_2)\ldots\}. \tag{21}$$

$n_i - N_i$ is always even. The trace in (21) can be expanded as a sum of terms where each term can be represented as follows. In the elementary square of the lattice located at the ith row and jth column ($i-j$ is odd) one puts a vertical bar if the term $\tilde{e}_j = \beta e_j$ is picked up in the matrix U or V and a horizontal bar if 1 is picked up. The operator $\varphi_m(n, N)$ is represented by a cross at the site located at the nth row and Nth column. Thus the sites of the lattice are grouped into clusters, all sites of a cluster having the same height. These clusters can be separated by boundaries drawn on the dual lattice. For example, the quantity

$$t = \mathrm{Tr}(\tilde{e}_4\tilde{e}_2\tilde{e}_5\varphi_{m'}(2)\tilde{e}_1\tilde{e}_4\tilde{e}_2\tilde{e}_3\varphi_m(3)\tilde{e}_3) \tag{22}$$

with

$$\tilde{e}_1 = \beta e_i = \mathrm{b}\overset{\mathrm{a}}{\underset{\mathrm{a}}{\diamond}}\mathrm{c} = \frac{(S_\mathrm{b}S_\mathrm{c})^{1/2}}{S_\mathrm{a}} \tag{23}$$

is represented in figure 4.

This graphical representation can be further simplified (figure 5). Each cluster of sites is represented by a point and points representing clusters with a common boundary are joined by a line. With each cluster are represented the index of operators φ_m which it contains. The contribution of such a graph to the trace is a sum of terms, one for each allowed assignment of height at points of the graph. The contribution of a height configuration is a product of terms, one for each cluster. Using the expression (23) of $\tilde{e}_i$, the factor for each cluster of height a is $S_a^{b_+-b_-}$ with b_+ (b_-) the number of boundaries surrounding (surrounded by) the cluster multiplied by the factors V_m^a/S^a of the operators φ_m in the cluster. The clusters containing all the points in the upper (lower) row take an additional factor S^a coming from definition (7) of the trace. If there are several operators φ_m in a cluster, the product of terms v_m^a/S^a can be expanded on the base v^a/S^a and we can restrict ourselves to the case where there is at most one φ_m by cluster. It gives the following rules for computing the trace. Links with one free end in figure 5 are successively removed using the rule

$$\ldots \overset{\alpha,m}{\cdot}\!\!-\!\!\!-\!\!\!-\!\!\overset{\beta,m'}{\cdot} \to \lambda_m \qquad \sum_{m''} C_{mm'}^{m''} \ldots \alpha_{m''} \tag{24}$$

with $C_{mm'}^{m''}$ given by

$$\frac{V_m^a V_{m'}^a}{S^a} = \sum_{m''} C_{mm'}^{m''} V_{m''}^a. \tag{25}$$

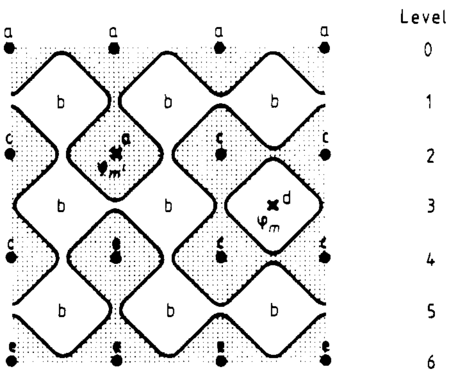

Figure 4. A cluster decomposition corresponding to t in formula (22).

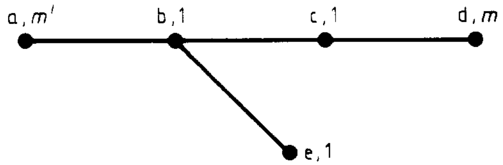

Figure 5. The graph corresponding to the cluster decomposition of figure 4.

V_m is normalised so that $C^0_{mm'} = \delta_{mm'}$. When there is only one point left, the trace is obtained by multiplying the coefficient of $m = 1$ by β^{-N} where $N+1$ is the number of rows of the lattice. In the example we are considering it gives

$$t = \beta^{-5}\lambda^3_m\delta_{mm'}. \tag{26}$$

The correlation function of the Φ_m operator therefore depends on the eigenvalues λ_m of the incidence matrix and on the coefficients $C^{m''}_{mm'}$. If there are only two operators φ_m, $\varphi_{m'}$, there is only one coefficient $C^0_{mm'} = \delta_{mm'}$. Thus, the correlation function $\langle\varphi_m, \varphi_{m'}\rangle$ is the same evaluated in the BC-SOS model provided we adjust β, λ_m, $\lambda_{m'}$ to be equal to the values of the original model. A non-zero result is obtained if we take one φ_m of each type in formula (20).

Under renormalisation, the BC-SOS model is expected to flow to a Coulomb gas (den Nijs 1983, Nienhuis 1984) with coupling constant $g = 4(1 - 1/h)$. In a modified version of the models where vacancies are allowed (Nienhuis *et al* 1979) another value of g, $g' = 4(1 + 1/h)$, can be reached. In this version, $\varphi_{h-1} = (-1)^{\sigma_1}$ is promoted from being a c number to an operator. In the renormalised model, $\varphi^1_m(n, N)$ becomes an electric operator that creates a charge $q_1 = (m-1)2/h$ at point (n, N) of the lattice and $\varphi^2_m(n, N)$ creates a charge $q_2 = (-m-1)2/h$. Electric neutrality is preserved because definition (7) of the trace introduces a charge $4/h$ at infinity (den Nijs 1984, Dotsenko and Fateev 1984). The correlation function $\langle\varphi^1_m(\boldsymbol{x})\varphi^2_m(\boldsymbol{x}+\boldsymbol{r})\rangle$ therefore behaves as

$$r^{(q_1q_2)/g} = \begin{cases} r^{(m^2-1)/h(h-1)} & \text{in the model without vacancies} \\ r^{(m^2-1)/h(h+1)} & \text{in the model with vacancies} \end{cases} \tag{27}$$

which yields

$$x_m = m^2 - 1/2h(h-1) \qquad \text{or} \qquad x_m = m^2 - 1/2h(h+1). \tag{28}$$

In the A_n case, these scaling dimensions had been derived by Huse (1984) from the exact solution of Andrews *et al.* In the three-state Potts model (D_4) case they had been obtained by den Nijs (1983) using a similar method.

3. Bimodule decomposition and fusion rules

In this section, we shall make some connection between the lattice operator algebra and the operator algebra in conformal theories. In particular we shall show that it is consistent to identify the TLJ algebra with the algebra of thermal operators. The total algebra then decomposes into bimodules with product rules that agree with the fusion rules of Belavin *et al* (1984) (BPZ).

In conformal theories, the operator algebra splits into representations of the left and right Virasoro algebras (Belavin *et al* 1984). They are labelled by two pairs of two integers $[r, s]$ and $[\bar{r}, \bar{s}]$, $1 \leq r, \bar{r} \leq h-1$, $1 \leq s, \bar{s} \leq h$ (h is an integer that characterises the theory) with $[r, s] = [h-r-h-s]$ and similarly for $\bar{r}, \bar{s}$. The product of operators in two different representations obey fusion rules which are basically the rules for multiplying representations of SU(2):

$$[r, s][r', s'] = \bigoplus_{\substack{|r-r'|+1 \leq r'' \leq |r+r'|-1 \\ |s-s'|+1 \leq s'' \leq |s-s'|-1}} [r'', s'']$$

and (29)

$$r'' = r - r' - 1 \qquad \text{mod } 2$$

$$s'' = s - s' - 1 \qquad \text{mod } 2$$

and similarly for $\bar{r}, \bar{s}$.

These rules indicate that the set of operators $s = \bar{s} = 1$, r, $\bar{r}$ odd, form an algebra: the thermal operator algebra. The total algebra considered as a vector space decomposes into modules over this algebra now labelled by s and $\bar{s}$:

$$(s, \bar{s}) = \bigoplus_{r, \bar{r}} [r, s][\bar{r}, \bar{s}]$$

$$r - s = 0 \qquad \text{mod } 2 \tag{30}$$

$$\bar{r} - \bar{s} = 0 \qquad \text{mod } 2.$$

The term 'module' simply means that $(s, \bar{s})$ is stable by multiplication with operators in the algebra $(1, 1)$.

In the lattice models, it is natural to identify $(1, 1)$ with the TLJ algebra. Let us now build molecules $[m]$ where m is an exponent of E_6.

Consider operators obtained by taking linear combinations of $\varphi_m(N)$ at any level N multiplied left or right by any possible polynomial in e_n. One such operator is, for example,

$$e_3\varphi_m(1)e_4e_5 + e_6e_5\varphi_m(5)e_6 + \varphi_m(9). \tag{31}$$

Let $[m]$ denote this set. It is by construction a module over $[1]$ of the TLJ algebra. For the scalar product defined by $\langle x|y\rangle = \mathrm{Tr}\, x^*y$, any operator in $[m]$ is orthogonal to any operator in $[m']$ for $m \neq m'$. It is sufficient for that to compute $\mathrm{Tr}(\varphi_m(0)w\varphi_{m'}$, $(n)w')$ with w and w' two monomials in e_1. Using the graphical method explained in § 2 one sees that the result is always proportional to $V_m^a V_{m'}^a$, which is null for $m \neq m'$ since $\{V_m\}$ is an orthogonal base. This in particular shows that $\langle\varphi_m\varphi_{m'}\rangle$ correlation functions of φ_m and $\varphi_{m'}$ vanish if $m \neq m'$. $\mathscr{D}$ can therefore be decomposed into

$$\mathscr{D} = \bigoplus_{m \in \text{exponents}} [m] \oplus (\text{other modules}). \tag{32}$$

To fully convince oneself that $[m]$ can be identified with $(m, \bar{m})$ in formula (30) one needs to verify that the BPZ fusion rules are satisfied, namely

$$[m][m'] = \bigoplus_{|m-m'|+1 \leq m'' \leq |m+m'|-1} [m''] \oplus (\text{other modules})$$

$$m'' - m - 1 = 0 \qquad \text{mod } 2 \tag{33}$$

$$m'' - m' - 1 = 0 \qquad \text{mod } 2.$$

The first line of the above equality is verified if

$$\mathrm{Tr}(\varphi_m(N)w\varphi_{m'}(N')w'\varphi_{m''}w'') \tag{34}$$

vanishes for w, w', w'' arbitrary monomial in e_1 and m'' not among the above values. The result is always proportional to

$$\sum_a \frac{V^a_m V^a_{m'} V^a_{m''}}{S^a}. \tag{35}$$

We therefore need to decompose the vector $W^a = \Sigma_a\ V^a_m V^a_{m'}/S^a$ on the base $\{V_{m''}\}$ and observe that the only values of m'' that occur are among those of formula (33). In the A_n case, for example,

$$v^a_m = \sin\frac{m\pi a}{n+1}$$

$$\frac{v^a_m v^a_{m'}}{S^a} = \sin\left(\frac{m\pi a}{n+1}\right)\sin\left(\frac{m'\pi a}{n+1}\right)\left[\sin\left(\frac{\pi a}{n+1}\right)\right]^{-1} = \sum_{|m'-m|+1\le m''\le|m'+m|-1} \sin\left(\frac{m''\pi a}{n+1}\right) \tag{36}$$

$$m''-m'-m-1=0 \qquad \mathrm{mod}\ 2.$$

This remains true for all Dynkin diagrams.

3.1. *Non-scalar operators in the Potts model case*

The question immediately arises: is the decomposition complete or are there other modules? Although we did not prove it, there are very likely not to be other modules for the A_n (RSOS) models but, as we shall now see in the case of D_4, there may be for the other Dynkin diagrams.

In the D_4 (Potts model) case, it is easy to build projectors that project on each module of the decomposition. Denote by O^+O^-2 the three external nodes of D_4. An outer action of $\mathbb{Z}_2\times\mathbb{Z}_3$ on $\mathscr{D}$ can be obtained as follows. Let T and G be the operators that exchange O^+, O^- and cyclically permute O^+O^-2 at every level of a path. For $x\in\mathscr{D}$ define

$$\begin{aligned} \varphi_T(x) &= TxT^{-1} \qquad & \varphi_T^2 &= 1 \\ \varphi_G(x) &= GxG^{-1} \qquad & \varphi_G^3 &= 1 \end{aligned} \tag{37}$$

$\{\varphi_T, \varphi_G\}$ determine an outer action of $\mathbb{Z}_2\otimes\mathbb{Z}_3$ on $\mathscr{D}$ and [1] can be characterised as the subalgebra of operators that remain unchanged by this action. We can now define the following projectors:

$$\begin{aligned} P_{[1]} &= [\tfrac{1}{3}(1+\varphi_G+\varphi_G^2)][\tfrac{1}{2}(1+\varphi_T)] \\ P_{[3^+]} &= [\tfrac{1}{3}(2-\varphi_G-\varphi_G^2)][\tfrac{1}{2}(1+\varphi_T)] \\ P_{[3^-]} &= [\tfrac{1}{3}(2-\varphi_G-\varphi_G^2)][\tfrac{1}{2}(1-\varphi_T)] \\ P_{[\Delta]} &= [\tfrac{1}{3}(1+\varphi_G+\varphi_G^2)][\tfrac{1}{2}(1-\varphi_T)]. \end{aligned} \tag{38}$$

As indicated $P_{[1]}$ projects on the [1] algebra, $P_{[3^+]}$ on the $[3^+]$ module obtained with $\varphi_{3^+}=2P_2-P_0-P_{0^+}\varphi_{[3^-]}$ on the $[3^-]$ module obtained with $\varphi_{3^-}=P_{0^+}-P_{0^-}$. $P_{[\Delta]}$

Table 1. List of known partition functions in terms of $A_1^{(1)}$ characters.

$k \geqslant 1$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	A_{k+1}
$k=4\rho,\ \rho \geqslant 1$	$\sum_{\substack{\lambda\ \mathrm{odd}=1\\ \lambda\neq 2\rho+1}}^{4\rho+1} (\lvert\chi_\lambda\rvert^2+2\lvert\chi_{2\rho+1}\rvert^2)\ \sum_{\lambda\ \mathrm{odd}=1}^{2\rho-1} (\chi_\lambda\chi^*_{4\rho+2-\lambda}+\mathrm{CC})$	$D_{2\rho+2}$
	$=\sum_{\lambda\ \mathrm{odd}=1}^{2\rho-1} (\lvert\chi_\lambda+\chi_{4\rho+2-\lambda}\rvert^2+2\lvert\chi_{2\rho+1}\rvert^2)$	
$k=4\rho-2,\ \rho\geqslant 2$	$\sum_{\lambda\ \mathrm{odd}=1}^{4\rho-1} (\lvert\chi_\lambda\rvert^2+\lvert\chi_{2\rho}\rvert^2)+\sum_{\lambda\ \mathrm{even}=2}^{2\rho-2} (\chi_\lambda\chi^*_{4\rho-\lambda}+\mathrm{CC})$	$D_{2\rho+1}$
$k+2=12$	$\lvert\chi_1+\chi_7\rvert^2+\lvert\chi_4+\chi_8\rvert^2+\lvert\chi_5+\chi_{11}\rvert^2$	E_6
$k+2=18$	$\lvert\chi_1+\chi_{17}\rvert^2+\lvert\chi_5+\chi_{13}\rvert^2+\lvert\chi_7+\chi_{11}\rvert^2+\lvert\chi_9\rvert^2$ $+[(\chi_3+\chi_{15})\chi_9^*+\mathrm{CC}]$	E_7
$k+2=30$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2+\lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	E_8

does not project on any previously built module. $[\Delta]$ would in fact have been obtained by repeating the above construction with

$$\varphi_\Delta(N)=P_{0^+0^-}+P_{0^-2}+P_{20^+}-P_{0^-0^+}-P_{20^-}-P_{0^+2} \qquad (39)$$

where $P_{0^+0^-}$ stands for $P_{\sigma_N=0^+,\ \sigma_{N+2}=2}\ldots$.

In the case of an arbitrary Dynkin diagram the module decomposition can be deduced from the analyses of modular invariant partition functions (Cappelli *et al* 1986). But various partition functions are listed in table 1; each term $X_sX_{\bar{s}}^*$ in the partition function corresponds to a module $(s,\bar{s})$. In the D_4 case, for example, the partition function is (Cardy 1986)

$$Z=X_1X_1^*+2X_3X_3^*+X_1X_5^*+X_5X_1^*+X_5X_5^*. \qquad (40)$$

We have the module decomposition

$$\mathscr{D}^+=[1]\oplus[3^+]\oplus[3^-]\oplus[5]\oplus[\Delta].$$

It is therefore consistent to identify

$$\begin{aligned}[5]\oplus[1]&=(1,1)+(5,5)\\ [3^+]&=(3,3)^+\\ [3^-]&=(3,3)^-\\ [\Delta]&=(1,5)\oplus(5,1).\end{aligned} \qquad (41)$$

φ_Δ should therefore be the chiral operator (den Nijs 1984).

In the case of other Dynkin diagrams, we have not yet been able to recover modules $(s,\bar{s})$ with $\bar{s}\neq s$. In the general case, we conjecture that the complete decomposition of the lattice algebra $\mathscr{D}$ considered as a module over the TLJ algebra coincides with the decomposition that can be derived from the partition functions in table 1.

Acknowledgment

I would like to thank V Jones for valuable discussions.

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